

# Affine nonexpansive operators, Attouch-Théra duality and the Douglas-Rachford algorithm

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March 30, 2016

*In tribute to Michel Théra on his 70th birthday*

## Abstract

The Douglas-Rachford splitting algorithm was originally proposed in 1956 to solve a system of linear equations arising from the discretization of a partial differential equation. In 1979, Lions and Mercier brought forward a very powerful extension of this method suitable to solve optimization problems.

In this paper, we revisit the original affine setting. We provide a powerful convergence result for finding a zero of the sum of two maximally monotone affine relations. As a by product of our analysis, we obtain results concerning the convergence of iterates of affine nonexpansive mappings as well as Attouch-Théra duality. Numerous examples are presented.

**2010 Mathematics Subject Classification:** Primary 47H05, 47H09, 49M27; Secondary 49M29, 49N15, 90C25.

**Keywords:** affine mapping, Attouch-Théra duality, Douglas-Rachford algorithm, linear convergence, maximally monotone operator, nonexpansive mapping, paramonotone operator, strong convergence, Toeplitz matrix, tridiagonal matrix.

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# 1 Introduction

Throughout this paper

$X$  is a real Hilbert space,

with inner product  $\langle \cdot, \cdot \rangle$  and induced norm  $\|\cdot\|$ . A central problem in optimization is to

$$\text{find } x \in X \text{ such that } 0 \in (A + B)x, \quad (1)$$

where  $A$  and  $B$  are maximally monotone operators on  $X$ ; see, e.g., [7], [14], [15], [17], [18], [34], [35], [33], [39], [40], and the references therein. As Lions and Mercier observed in their landmark paper [29], one may iteratively solve the *sum problem* (1) by the celebrated *Douglas-Rachford splitting algorithm* (see also [24]). This algorithm proceeds by iterating the operator  $T = \text{Id} - J_A + J_B R_A$ ; the sequence  $(J_A T^n x)_{n \in \mathbb{N}}$  converges to a solution of (1) (see Section 5 for details). The Douglas-Rachford algorithm was originally proposed in 1956 by Douglas and Rachford [21]. It can be viewed as a method for solving a system of linear equations where the underlying coefficient matrix is positive definite. The far-reaching extension to optimization provided by Lions and Mercier [29] is not at all obvious (for the sake of completeness, we sketch this connection in the Appendix).

In this paper, we concentrate on the affine setting. In the original setting considered by Douglas and Rachford, the operators  $A$  and  $B$  correspond to positive definite matrices. We extend this result in various directions. Indeed, we obtain *strong convergence* in possibly *infinite-dimensional* Hilbert space; the operators  $A$  and  $B$  may be *affine maximally monotone relations*, and we also identify the *limit*. The remainder of this paper is organized as follows. In Section 2, we provide several results which will be useful in the derivation of the main results. A new characterization of strongly convergent iterations of affine nonexpansive operators (Theorem 3.3) is presented in Section 3. We also discuss when the convergence is linear. In Section 4, we obtain new results, which are formulated using the Douglas-Rachford operator, on the relative geometry of the primal and dual (in the sense of Attouch-Théra duality) solutions to (1). The main algorithmic result (Theorem 5.1) is derived in Section 5. It provides precise information on the behaviour of the Douglas-Rachford algorithm in the affine case. Numerous examples are presented in Section 6 where we also pay attention to the tridiagonal Toeplitz matrices and Kronecker products. In the Appendix, we sketch the connection between the historical Douglas-Rachford algorithm and the powerful extension provided by Lions and Mercier.

Finally, the notation we employ is quite standard and follows largely [7]. Let  $C$  be a nonempty closed convex subset of  $X$ . We use  $N_C$  and  $P_C$  to denote the *normal cone operator* and the *projector* associated with  $C$ , respectively. Let  $Y$  be a Banach space. We shall use  $\mathcal{B}(Y)$  to denote the set of *bounded linear operators* on  $Y$ . Let  $L \in \mathcal{B}(Y)$ . The *operator norm* of  $L$  is  $\|L\| = \sup_{\|y\| \leq 1} \|Ly\|$ . Further notation is developed as necessary during the course of this paper.

## 2 Auxiliary results

In this section, we collect various results that will be useful in the sequel.

Suppose that  $T : X \rightarrow X$ . Then  $T$  is *nonexpansive* if

$$(\forall x \in X)(\forall y \in X) \quad \|Tx - Ty\| \leq \|x - y\|; \quad (2)$$

$T$  is *firmly nonexpansive* if

$$(\forall x \in X)(\forall y \in X) \quad \|Tx - Ty\|^2 + \|(\text{Id} - T)x - (\text{Id} - T)y\|^2 \leq \|x - y\|^2; \quad (3)$$

$T$  is *asymptotically regular* if

$$(\forall x \in X) \quad T^n x - T^{n+1} x \rightarrow 0. \quad (4)$$

**Fact 2.1.** *Let  $T : X \rightarrow X$ . Then*

$$\left. \begin{array}{l} T \text{ firmly nonexpansive} \\ \text{Fix } T \neq \emptyset \end{array} \right\} \Rightarrow T \text{ asymptotically regular.} \quad (5)$$

*Proof.* See [16, Corollary 1.1] or [7, Corollary 5.16(ii)]. ■

**Fact 2.2.** *Let  $L : X \rightarrow X$  be linear and nonexpansive, and let  $x \in X$ . Then*

$$L^n x \rightarrow P_{\text{Fix } L} x \Leftrightarrow L^n x - L^{n+1} x \rightarrow 0. \quad (6)$$

*Proof.* See [2, Proposition 4], [3, Theorem 1.1], [8, Theorem 2.2] or [7, Proposition 5.27]. (We mention in passing that in [2, Proposition 4] the author proved the result for general odd nonexpansive mappings in Hilbert spaces and in [3, Theorem 1.1], the authors generalize the result to Banach spaces.) ■

**Definition 2.3.** *Let  $Y$  be a real Banach space, let  $(y_n)_{n \in \mathbb{N}}$  be a sequence in  $Y$  and let  $y_\infty \in Y$ . Then  $(y_n)_{n \in \mathbb{N}}$  converges to  $y_\infty$ , denoted  $y_n \rightarrow y_\infty$ , if  $\|y_n - y_\infty\| \rightarrow 0$ .  $(y_n)_{n \in \mathbb{N}}$  converges  $\mu$ -linearly to  $y_\infty$  if  $\mu \in [0, 1[$  and there exists  $M \geq 0$  such that<sup>1</sup>*

$$(\forall n \in \mathbb{N}) \quad \|y_n - y_\infty\| \leq M\mu^n. \quad (7)$$

$(y_n)_{n \in \mathbb{N}}$  converges linearly to  $y_\infty$  if there exists  $\mu \in [0, 1[$  and  $M \geq 0$  such that (7) holds.

**Example 2.4 (convergence vs. pointwise convergence of bounded linear operators).** *Let  $Y$  be a real Banach space, let  $(L_n)_{n \in \mathbb{N}}$  be a sequence in  $\mathcal{B}(Y)$ , and let  $L_\infty \in \mathcal{B}(Y)$ . Then one says:*

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<sup>1</sup> By [10, Remark 3.7], this is equivalent to  $(\exists M > 0)(\exists N \in \mathbb{N})(\forall n \geq N) \|y_n - y_\infty\| \leq M\mu^n$ .

- (i)  $(L_n)_{n \in \mathbb{N}}$  converges or converges uniformly to  $L_\infty$  in  $\mathcal{B}(Y)$  if  $L_n \rightarrow L_\infty$  (in  $\mathcal{B}(Y)$ ).
- (ii)  $(L_n)_{n \in \mathbb{N}}$  converges pointwise to  $L_\infty$  if  $(\forall y \in Y) L_n y \rightarrow L_\infty y$  (in  $Y$ ).

**Remark 2.5.** It is easy to see that the convergence of a sequence of bounded linear operators implies pointwise convergence; however, the converse is not true (see, e.g., [27, Example 4.9-2]).

**Lemma 2.6.** Let  $Y$  be a real Banach space, let  $(L_n)_{n \in \mathbb{N}}$  be a sequence in  $\mathcal{B}(Y)$ , let  $L_\infty \in \mathcal{B}(Y)$ , and let  $\mu \in ]0, 1[$ . Then

$$(\forall y \in Y) L_n y \rightarrow L_\infty y \text{ } \mu\text{-linearly (in } Y) \Leftrightarrow L_n \rightarrow L_\infty \text{ } \mu\text{-linearly (in } \mathcal{B}(Y)). \quad (8)$$

*Proof.* Let  $y \in Y$ . “ $\Rightarrow$ ”: Because  $L_n y \rightarrow L_\infty y$   $\mu$ -linearly, there exists  $M_y \geq 0$  such that  $(\forall n \in \mathbb{N}) \| (L_n - L_\infty)y \| \leq \mu^n M_y$ ; equivalently,

$$\left\| \left( \frac{L_n - L_\infty}{\mu^n} \right) y \right\| = \frac{\| (L_n - L_\infty)y \|}{\mu^n} \leq M_y. \quad (9)$$

It follows from the Uniform Boundedness Principle (see, e.g., [27, 4.7-3]) applied to the sequence  $((L_n - L_\infty)/\mu^n)_{n \in \mathbb{N}}$  that  $(\exists M \geq 0)(\forall n \in \mathbb{N}) \|(L_n - L_\infty)/\mu^n\| \leq M$ ; equivalently,  $\|L_n - L_\infty\| \leq M\mu^n$ , as required. “ $\Leftarrow$ ”: Since  $L_n \rightarrow L_\infty$   $\mu$ -linearly, we have  $(\exists M \geq 0)(\forall n \in \mathbb{N}) \|L_n - L_\infty\| \leq M\mu^n$ . Therefore,  $(\forall n \in \mathbb{N}) \|L_n y - L_\infty y\| \leq \|L_n - L_\infty\| \|y\| \leq M\mu^n \|y\|$ . ■

**Lemma 2.7.** Suppose that  $X$  is finite-dimensional, let  $(L_n)_{n \in \mathbb{N}}$  be a sequence of linear nonexpansive operators on  $X$  and let  $L_\infty : X \rightarrow X$ . Then the following are equivalent:

- (i)  $(\forall x \in X) L_n x \rightarrow L_\infty x$ .
- (ii)  $L_n \rightarrow L_\infty$  pointwise (in  $X$ ), and  $L_\infty$  is linear and nonexpansive.
- (iii)  $L_n \rightarrow L_\infty$  (in  $\mathcal{B}(X)$ ).

*Proof.* The implications “(i) $\Rightarrow$ (ii)” and “(iii) $\Rightarrow$ (i)” are easy to verify. “(ii) $\Rightarrow$ (iii)”: Suppose that  $(x_n)_{n \in \mathbb{N}}$  is a sequence in  $X$  such that  $(\forall n \in \mathbb{N}) \|x_n\| = 1$  and

$$\|L_n - L_\infty\| - \|L_n x_n - L_\infty x_n\| \rightarrow 0. \quad (10)$$

We can and do assume that  $x_n \rightarrow x_\infty$ . Since  $L_\infty$  and  $(L_n)_{n \in \mathbb{N}}$  are linear and nonexpansive, we have  $\|L_\infty\| \leq 1$  and  $(\forall n \in \mathbb{N}) \|L_n\| \leq 1$ . Using the triangle inequality, we have  $\|L_n x_n - L_\infty x_n\| = \|(L_n - L_\infty)(x_n - x_\infty) + (L_n - L_\infty)x_\infty\| \leq \|L_n - L_\infty\| \|x_n - x_\infty\| + \|(L_n - L_\infty)x_\infty\| \leq 2\|x_n - x_\infty\| + \|(L_n - L_\infty)x_\infty\| \rightarrow 0 + 0 = 0$ . Now combine with (10). ■

**Corollary 2.8.** Suppose that  $X$  is finite-dimensional, let  $L : X \rightarrow X$  be linear, and let  $L_\infty : X \rightarrow X$  be such that  $L^n \rightarrow L_\infty$  pointwise. Then  $L^n \rightarrow L_\infty$  linearly.

*Proof.* Combine Lemma 2.7 and [5, Theorem 2.12(i)]. ■

### 3 Iterating an affine nonexpansive operator

We begin with a simple yet useful result.

**Theorem 3.1.** *Let  $L: X \rightarrow X$  be linear, let  $b \in X$ , set  $T: X \rightarrow X: x \mapsto Lx + b$ , and suppose that  $\text{Fix } T \neq \emptyset$ . Let  $x \in X$ . Then the following hold:*

- (i)  $b \in \text{ran}(\text{Id} - L)$ .
- (ii)  $(\forall n \in \mathbb{N}) \quad T^n x = L^n x + \sum_{k=0}^{n-1} L^k b$ .

*Proof.* (i):  $\text{Fix } T \neq \emptyset \Leftrightarrow (\exists y \in X) \ y = Ly + b \Leftrightarrow b \in \text{ran}(\text{Id} - L)$ . (ii): We prove this by induction (see also [11, Theorem 3.2(ii)]). When  $n = 0$  or  $n = 1$  the conclusion is obviously true. Now suppose that, for some  $n \in \mathbb{N}$ ,

$$T^n x = L^n x + \sum_{k=0}^{n-1} L^k b. \quad (11)$$

Then  $T^{n+1}x = T(T^n x) = T(L^n x + \sum_{k=0}^{n-1} L^k b) = L(L^n x + \sum_{k=0}^{n-1} L^k b) + b = L^{n+1}x + \sum_{k=0}^n L^k b$ . ■

Let  $S$  be a nonempty closed convex subset of  $X$  and let  $w \in X$ . We recall the following useful translation formula (see, e.g., [7, Proposition 3.17]):

$$(\forall x \in X) \quad P_{w+S}x = w + P_S(x - w). \quad (12)$$

**Lemma 3.2.** *Let  $L: X \rightarrow X$  be linear and nonexpansive, let  $b \in X$ , set  $T: X \rightarrow X: x \mapsto Lx + b$ , and suppose that  $\text{Fix } T \neq \emptyset$ . Then there exists a point  $a \in X$  such that  $b = a - La$  and*

$$(\forall x \in X) \quad Tx = L(x - a) + a. \quad (13)$$

Moreover, the following hold:

- (i)  $\text{Fix } T = a + \text{Fix } L$ .
- (ii)  $(\forall x \in X) \quad P_{\text{Fix } T}x = a + P_{\text{Fix } L}(x - a) = P_{(\text{Fix } L)^\perp}a + P_{\text{Fix } L}x$ .
- (iii)  $(\forall n \in \mathbb{N})(\forall x \in X) \quad T^n x = a + L^n(x - a)$ .

*Proof.* The existence of  $a$  and (13) follows from Theorem 3.1 and the linearity of  $L$ . (i): Let  $y \in X$ . Then  $y \in \text{Fix } T \Leftrightarrow y - a \in \text{Fix } L \Leftrightarrow y \in a + \text{Fix } L$ . (ii): The first identity follows from combining (i) and (12). It follows from, e.g., [7, Corollary 3.22(ii)] that  $a + P_{\text{Fix } L}(x - a) = a + P_{\text{Fix } L}x - P_{\text{Fix } L}a = P_{(\text{Fix } L)^\perp}a + P_{\text{Fix } L}x$ . (iii): By telescoping, we have

$$\sum_{k=0}^{n-1} L^k b = \sum_{k=0}^{n-1} L^k(a - La) = a - L^n a. \quad (14)$$

Consequently, Theorem 3.1(ii) and (14) yield  $T^n x = L^n x + a - L^n a = a + L^n(x - a)$ . ■

The following result extends Fact 2.2 from the linear to the affine case.

**Theorem 3.3.** *Let  $L : X \rightarrow X$  be linear and nonexpansive, let  $b \in X$ , set  $T : X \rightarrow X : x \mapsto Lx + b$ , and suppose that  $\text{Fix } T \neq \emptyset$ . Then the following are equivalent:*

- (i)  $L$  is asymptotically regular.
- (ii)  $L^n \rightarrow P_{\text{Fix } L}$  pointwise.
- (iii)  $T^n \rightarrow P_{\text{Fix } T}$  pointwise.
- (iv)  $T$  is asymptotically regular.

*Proof.* Let  $x \in X$ . “(i) $\Leftrightarrow$ (ii)”: This is Fact 2.2. “(ii) $\Rightarrow$ (iii)”: In view of Lemma 3.2(iii)&(ii) we have  $T^n x = L^n(x - a) + a \rightarrow P_{\text{Fix } L}(x - a) + a = P_{\text{Fix } T}x$ . “(iii) $\Rightarrow$ (iv)”:  $T^n x - T^{n+1}x \rightarrow P_{\text{Fix } T}x - P_{\text{Fix } T}x = 0$ . “(iv) $\Rightarrow$ (i)”: Using Lemma 3.2(iii) we have  $L^n x - L^{n+1}x = T^n(x + a) - T^{n+1}(x + a) \rightarrow 0$ . ■

We now turn to linear convergence.

**Lemma 3.4.** *Suppose that  $X$  is finite-dimensional, and let  $L : X \rightarrow X$  be linear and nonexpansive. Then the following are equivalent:*

- (i)  $L$  is asymptotically regular.
- (ii)  $L^n \rightarrow P_{\text{Fix } L}$  pointwise (in  $X$ ).
- (iii)  $L^n \rightarrow P_{\text{Fix } L}$  (in  $\mathcal{B}(X)$ ).
- (iv)  $L^n \rightarrow P_{\text{Fix } L}$  linearly pointwise (in  $X$ ).
- (v)  $L^n \rightarrow P_{\text{Fix } L}$  linearly (in  $\mathcal{B}(X)$ ).

*Proof.* “(i) $\Leftrightarrow$ (ii)”: This follows from Fact 2.2. “(ii) $\Leftrightarrow$ (iii)”: Combine Lemma 2.7 and Fact 2.2. “(iii) $\Rightarrow$ (v)”: Apply Corollary 2.8 with  $L_\infty$  replaced by  $P_{\text{Fix } L}$ . “(v) $\Rightarrow$ (iii)”: This is obvious. “(iv) $\Leftrightarrow$ (v)”: Apply Lemma 2.6 to the sequence  $(L^n)_{n \in \mathbb{N}}$  and use Fact 2.2. ■

**Theorem 3.5.** *Let  $L : X \rightarrow X$  be linear and nonexpansive, let  $b \in X$ , set  $T : X \rightarrow X : x \mapsto Lx + b$  and let  $\mu \in ]0, 1[$ . Then the following are equivalent:*

- (i)  $T^n \rightarrow P_{\text{Fix } T}$   $\mu$ -linearly pointwise (in  $X$ ).
- (ii)  $L^n \rightarrow P_{\text{Fix } L}$   $\mu$ -linearly pointwise (in  $X$ ).
- (iii)  $L^n \rightarrow P_{\text{Fix } L}$   $\mu$ -linearly (in  $\mathcal{B}(X)$ ).

*Proof.* “(i) $\Leftrightarrow$ (ii)”: It follows from Lemma 3.2(iii)&(ii) that  $T^n x - P_{\text{Fix } T}x = a + L^n(x - a) - (a + P_{\text{Fix } L}(x - a)) = L^n(x - a) - P_{\text{Fix } L}(x - a) \rightarrow 0$ , by Fact 2.2. “(ii) $\Leftrightarrow$ (iii)”: Combine Lemma 2.6 and Fact 2.2. ■

**Corollary 3.6.** Suppose that  $X$  is finite-dimensional. Let  $L : X \rightarrow X$  be linear, nonexpansive and asymptotically regular, let  $b \in X$ , set  $T : X \rightarrow X : x \mapsto Lx + b$  and suppose that  $\text{Fix } T \neq \emptyset$ . Then  $T^n \rightarrow P_{\text{Fix } T}$  pointwise linearly.

*Proof.* It follows from Fact 2.2 that  $L^n \rightarrow P_{\text{Fix } L}$  pointwise. Consequently, by Corollary 2.8,  $L^n \rightarrow P_{\text{Fix } L}$  linearly. Now apply Theorem 3.5. ■

## 4 Attouch-Théra duality

Recall that a possibly set-valued operator  $A : X \rightrightarrows X$  is *monotone* if for any two points  $(x, u)$  and  $(y, v)$  in the *graph* of  $A$ , denoted  $\text{gra } A$ , we have  $\langle x - y, u - v \rangle \geq 0$ ;  $A$  is *maximally monotone* if there is no proper extension of  $\text{gra } A$  that preserves the monotonicity of  $A$ . The *resolvent*<sup>2</sup> of  $A$ , denoted by  $J_A$ , is defined by  $J_A = (\text{Id} + A)^{-1}$  while the *reflected resolvent* of  $A$  is  $R_A = 2J_A - \text{Id}$ . In the following, we assume that

$A : X \rightrightarrows X$  and  $B : X \rightrightarrows X$  are maximally monotone.

The *Attouch-Théra* (see [1]) dual pair to the primal pair  $(A, B)$  is the pair<sup>3</sup>  $(A^{-1}, B^{-\circledast})$ . The *primal* problem associated with  $(A, B)$  is to

$$\text{find } x \in X \text{ such that } 0 \in Ax + Bx, \quad (15)$$

and its Attouch-Théra *dual* problem is to

$$\text{find } x \in X \text{ such that } 0 \in A^{-1}x + B^{-\circledast}x. \quad (16)$$

We shall use  $Z$  and  $K$  to denote the sets of primal and dual solutions of (15) and (16) respectively, i.e.,

$$Z = Z_{(A,B)} = (A + B)^{-1}(0) \quad \text{and} \quad K = K_{(A,B)} = (A^{-1} + B^{-\circledast})^{-1}(0). \quad (17)$$

The *Douglas-Rachford* operator for the ordered pair  $(A, B)$  (see [29]) is defined by

$$T_{\text{DR}} = T_{\text{DR}}(A, B) = \text{Id} - J_A + J_B R_A = \frac{1}{2}(\text{Id} + R_B R_A). \quad (18)$$

We recall that  $C : X \rightrightarrows X$  is *paramonotone*<sup>4</sup> if it is monotone and  $(\forall (x, u) \in \text{gra } C) (\forall (y, v) \in \text{gra } C)$  we have

$$\left. \begin{array}{l} (x, u) \in \text{gra } C \\ (y, v) \in \text{gra } C \\ \langle x - y, u - v \rangle = 0 \end{array} \right\} \Rightarrow \{(x, v), (y, u)\} \subseteq \text{gra } C. \quad (19)$$

<sup>2</sup>It is well-known for a maximally monotone operator  $A : X \rightrightarrows X$  that  $J_A$  is firmly nonexpansive and  $R_A$  is nonexpansive (see, e.g. [7, Corollary 23.10(i) and (ii)]).

<sup>3</sup>Let  $A : X \rightrightarrows X$ . Then  $A^{\circledast} = (-\text{Id}) \circ A \circ (-\text{Id})$  and  $A^{-\circledast} = (A^{-1})^{\circledast} = (A^{\circledast})^{-1}$ .

<sup>4</sup>For a detailed discussion on paramonotone operators we refer the reader to [26].

**Example 4.1.** Let  $f : X \rightarrow ]-\infty, +\infty]$  be proper, convex and lower semicontinuous. Then  $\partial f$  is paramonotone by [26, Proposition 2.2] (or by [7, Example 22.3(i)]).

**Example 4.2.** Suppose that  $X = \mathbb{R}^2$  and that  $A : \mathbb{R}^2 \rightarrow \mathbb{R}^2 : (x, y) \mapsto (y, -x)$ . Then one can easily verify that  $A$  and  $-A$  are maximally monotone but not paramonotone by [26, Section 3] (or [13, Theorem 4.9]).

**Fact 4.3.** The following hold:

- (i)  $T_{\text{DR}}$  is firmly nonexpansive.
- (ii)  $\text{zer}(A + B) = J_A(\text{Fix } T_{\text{DR}})$ .

If  $A$  and  $B$  are paramonotone, then we have additionally:

- (iii)  $\text{Fix } T_{\text{DR}} = Z + K$ .
- (iv)  $(K - K) \perp (Z - Z)$ .

*Proof.* (i): See [29, Lemma 1], [23, Corollary 4.2.1], or [7, Proposition 4.21(ii)]. (ii): See [19, Lemma 2.6(iii)] or [7, Proposition 25.1(ii)]. (iii): See [6, Corollary 5.5(iii)]. (iv): See [6, Corollary 5.5(iv)]. ■

**Lemma 4.4.** Suppose that  $A$  and  $B$  are paramonotone. Let  $k \in K$  be such that  $(\forall z \in Z) J_A(z + k) = P_Z(z + k)$ . Then  $k \in (Z - Z)^\perp$ .

*Proof.* By Fact 4.3(iii),  $\text{Fix } T_{\text{DR}} = Z + K$ . Let  $z_1$  and  $z_2$  be in  $Z$ . It follows from [6, Theorem 4.5] that  $(\forall z \in Z) J_A(z + k) = z$ . Therefore,

$$(\forall i \in \{1, 2\}) \quad z_i + k \in \text{Fix } T_{\text{DR}} \text{ and } z_i = J_A(z_i + k) = P_Z(z_i + k). \quad (20)$$

Furthermore, the Projection Theorem (see, e.g., [7, Theorem 3.14]) yields

$$\langle k, z_1 - z_2 \rangle = \langle z_1 + k - z_1, z_1 - z_2 \rangle = \langle z_1 + k - P_Z(z_1 + k), P_Z(z_1 + k) - z_2 \rangle \geq 0. \quad (21)$$

On the other hand, interchanging the roles of  $z_1$  and  $z_2$  yields  $\langle k, z_2 - z_1 \rangle \geq 0$ . Altogether,  $\langle k, z_1 - z_2 \rangle = 0$ . ■

The next result relates the Douglas-Rachford operator to orthogonal properties of primal and dual solutions.

**Theorem 4.5.** Suppose that  $A$  and  $B$  are paramonotone. Then the following are equivalent:

- (i)  $J_A P_{\text{Fix } T_{\text{DR}}} = P_Z$ .
- (ii)  $J_A|_{\text{Fix } T_{\text{DR}}} = P_Z|_{\text{Fix } T_{\text{DR}}}$ .
- (iii)  $K \perp (Z - Z)$ .



*Proof.* “(i)⇒(ii)”: This is obvious. “(ii)⇒(iii)”: Let  $k \in K$  and let  $z \in Z$ . Then  $\text{Fix } T_{\text{DR}} = Z + K$  by Fact 4.3(iii); hence,  $z + k \in \text{Fix } T_{\text{DR}}$ . Therefore  $J_A(z + k) = P_Z(z + k)$ . Now apply Lemma 4.4. “(iii)⇒(i)”: This follows from [6, Theorem 6.7(ii)]. ■

**Corollary 4.6.** Let <sup>5</sup>  $U$  be a closed affine subspace of  $X$ , suppose that  $A = N_U$  and that  $B$  is paramonotone such that  $Z \neq \emptyset$ . Then the following hold<sup>6</sup>:

- (i)  $Z = U \cap (B^{-1}(\text{par } U)^\perp) \subseteq U$ .
- (ii)  $(\forall z \in Z) K = (-Bz) \cap (\text{par } U)^\perp \subseteq (\text{par } U)^\perp$ .
- (iii)  $K \perp (Z - Z)$ .
- (iv)  $J_A P_{\text{Fix } T_{\text{DR}}} = P_U P_{\text{Fix } T_{\text{DR}}} = P_Z$ .

*Proof.* Since  $A = N_C = \partial \iota_C$ , it is paramonotone by Example 4.1. (i): Let  $x \in X$ . Then  $x \in Z \Leftrightarrow 0 \in Ax + Bx = (\text{par } U)^\perp + Bx \Leftrightarrow [x \in U \text{ and there exists } y \in X \text{ such that } y \in (\text{par } U)^\perp \text{ and } y \in Bx] \Leftrightarrow [x \in U \text{ and there exists } y \in X \text{ such that } x \in B^{-1}y \text{ and } y \in (\text{par } U)^\perp] \Leftrightarrow x \in U \cap B^{-1}((\text{par } U)^\perp)$ . (ii): Let  $z \in Z$ . Applying [6, Remark 5.4] to  $(A^{-1}, B^{-\odot})$  yields  $K = (-Bz) \cap (Az) = (-Bz) \cap (\text{par } U)^\perp$ . (iii): By (i)  $Z - Z \subseteq U - U = \text{par } U$ . Now use (ii). (iv): Combine (iii) and Theorem 4.5. ■

Using [6, Proposition 2.10] we have

$$\text{zer } A \cap \text{zer } B \neq \emptyset \Leftrightarrow 0 \in K. \quad (22)$$

**Theorem 4.7.** Suppose that  $A$  and  $B$  are paramonotone and that  $\text{zer } A \cap \text{zer } B \neq \emptyset$ . Then the following hold:

- (i)  $Z = (\text{zer } A) \cap (\text{zer } B)$  and  $0 \in K$ .
- (ii)  $J_A P_{\text{Fix } T_{\text{DR}}} = P_Z$ .
- (iii)  $K \perp (Z - Z)$ .

If, in addition,  $A$  or  $B$  is single-valued, then we also have:

- (iv)  $K = \{0\}$ .
- (v)  $\text{Fix } T_{\text{DR}} = (\text{zer } A) \cap (\text{zer } B)$ .

*Proof.* (i): Since  $\text{zer } A \cap \text{zer } B \neq \emptyset$ , it follows from (22) that  $0 \in K$ . Now apply [6, Remark 5.4] to get  $Z = A^{-1}(0) \cap B^{-1}(0) = (\text{zer } A) \cap (\text{zer } B)$ . (ii): This is [6, Corollary 6.8]. (iii): Combine (22) and Fact 4.3(iv). (iv): Let  $C \in \{A, B\}$  be single-valued. Using (i) we have  $Z \subseteq \text{zer } C$ . Suppose that  $C = A$  and let  $z \in Z$ . We use [6, Remark 5.4] applied to  $(A^{-1}, B^{-\odot})$  to learn that  $K = (Az) \cap (-Bz)$ . Therefore  $\{0\} \subseteq K \subseteq Az \subseteq A(\text{zer } A) = \{0\}$ . A similar argument applies if  $C = B$ . (v): Combine Fact 4.3(iii) with (i) & (iv). ■

<sup>5</sup>Let  $C$  be nonempty closed convex subset of  $X$ . Then  $J_{N_C} = P_C$  by, e.g., [7, Example 23.4].

<sup>6</sup>Suppose that  $U$  is a closed affine subspace of  $X$ . We use  $\text{par } U$  to denote the *parallel space* of  $U$  defined by  $\text{par } U = U - U$ .

**Remark 4.8.** The conclusion of Theorem 4.7(i) generalizes the setting of convex feasibility problems. Indeed, suppose that  $A = N_U$  and  $B = N_V$ , where  $U$  and  $V$  are nonempty closed convex subsets of  $X$  such that  $U \cap V \neq \emptyset$ . Then  $Z = U \cap V = \text{zer } A \cap \text{zer } B$ .

The assumptions that  $A$  and  $B$  are paramonotone are critical in the conclusion of Theorem 4.7(i) as we illustrate now.

**Example 4.9.** Suppose that  $X = \mathbb{R}^2$ , that  $U = \mathbb{R} \times \{0\}$ , that  $A = N_U$  and that  $B : \mathbb{R}^2 \rightarrow \mathbb{R}^2 : (x, y) \mapsto (-y, x)$ , is the counterclockwise rotator in the plane by  $\pi/2$ . Then one verifies that  $\text{zer } A = U$ ,  $\text{zer } B = \{(0, 0)\}$ ,  $Z = \text{zer}(A + B) = U$ ; however  $(\text{zer } A) \cap (\text{zer } B) = \{(0, 0)\} \neq U = Z$ . Note that  $A$  is paramonotone by Example 4.1 while  $B$  is not paramonotone by Example 4.2.

In view of (22) and Theorem 4.7(ii), when  $A$  and  $B$  are paramonotone, we have the implication  $0 \in K \Rightarrow J_A P_{\text{Fix } T_{\text{DR}}} = P_Z$ . However the converse implication is not true, as we show in the next example.

**Example 4.10.** Suppose that  $a \in X \setminus \{0\}$ , that  $A = \text{Id} - 2a$  and that  $B = \text{Id}$ . Then  $Z = \{a\}$ ,  $(A^{-1}, B^{-\odot}) = (\text{Id} + 2a, \text{Id})$ , hence  $K = \{-a\}$ ,  $Z - Z = \{0\}$  and therefore  $K \perp (Z - Z)$  which implies that  $J_A P_{\text{Fix } T_{\text{DR}}} = P_Z$  by Theorem 4.5, but  $0 \notin K$ .

If neither  $A$  nor  $B$  is single-valued, then the conclusion of Theorem 4.7(iv)&(v) may fail as we now illustrate.

**Example 4.11.** Suppose that  $X = \mathbb{R}^2$ , that  $U = \mathbb{R} \times \{0\}$ , that<sup>7</sup>  $V = \text{ball}((0, 1); 1)$ , that  $A = N_U$  and that  $B = N_V$ . By [6, Example 2.7]  $Z = U \cap V = \{(0, 0)\}$  and  $K = N_{\overline{U-V}}(0) = \mathbb{R}_+ \cdot (0, 1) \neq \{(0, 0)\}$ . Therefore  $\text{Fix } T_{\text{DR}} = \mathbb{R}_+ \cdot (0, 1) \neq \{(0, 0)\} = U \cap V = \text{zer } A \cap \text{zer } B$ .

Recall that the Passty's parallel sum (see e.g., [32] or [7, Section 24.4]) is defined by

$$A \square B = (A^{-1} + B^{-1})^{-1}. \quad (23)$$

In view of (17) and (23), one readily verifies that

$$K = (A \square B^{\odot})(0). \quad (24)$$

**Lemma 4.12.** Suppose that  $B : X \rightrightarrows X$  is linear<sup>8</sup>. Then the following hold:

- (i)  $B^{\odot} = B$  and  $B^{-\odot} = B^{-1}$ .

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<sup>7</sup>Let  $u \in X$  and let  $r > 0$ . We use  $\text{ball}(u; r)$  to denote the closed ball in  $X$  centred at  $u$  with radius  $r$ . We also use  $\mathbb{R}_+$  to denote the set of nonnegative real numbers  $[0, +\infty[$ .

<sup>8</sup>  $A : X \rightrightarrows X$  is a linear relation if  $\text{gra } A$  is a linear subspace of  $X \times X$ .

- (ii)  $(A^{-1}, B^{-\circ}) = (A^{-1}, B^{-1})$ .
- (iii)  $K = (A \square B)(0)$ .

*Proof.* This is straightforward from the definitions. ■

Let  $f : X \rightarrow ]-\infty, +\infty]$  be proper, convex and lower semicontinuous. In the following we make use of the well-known identity<sup>9</sup> (see, e.g., [7, Corollary 16.24]):

$$(\partial f)^{-1} = \partial f^*. \quad (25)$$

**Corollary 4.13 (subdifferential operators).** *Let  $f : X \rightarrow ]-\infty, +\infty]$  and  $g : X \rightarrow ]-\infty, +\infty]$  be proper, convex and lower semicontinuous. Suppose that  $A = \partial f$  and that  $B = \partial g$ . Then the following hold<sup>10</sup>:*

- (i)  $Z = (\partial f^* \square \partial g^*)(0)$ .
- (ii)  $K = (\partial f \square \partial g^\vee)(0)$ .
- (iii) Suppose that<sup>11</sup>  $\text{Argmin } f \cap \text{Argmin } g \neq \emptyset$ . Then  $Z = \partial f^*(0) \cap \partial g^*(0)$ .
- (iv) Suppose that<sup>12</sup>  $0 \in \text{sri}(\text{dom } f - \text{dom } g)$ . Then<sup>13</sup>  $Z = \partial(f^* \square g^*)(0)$ .
- (v) Suppose that  $0 \in \text{sri}(\text{dom } f^* + \text{dom } g^*)$ . Then  $K = \partial(f \square g^\vee)(0)$ .

*Proof.* Note that  $A$  and  $B$  are paramonotone by Example 4.1. (i): Using (25) and (23) we have  $Z = (A + B)^{-1}(0) = (((\partial f)^{-1})^{-1} + ((\partial g)^{-1})^{-1})^{-1}(0) = ((\partial f^*)^{-1} + (\partial g^*)^{-1})^{-1}(0) = (\partial f^* \square \partial g^*)(0)$ . (ii): Observe that  $(\partial g)^{-\circ} = ((\partial g)^\circ)^{-1} = (\partial g^\vee)^{-1}$ . Therefore using (23) we have  $K = ((\partial f)^{-1} + ((\partial g)^\circ)^{-1})^{-1}(0) = ((\partial f)^{-1} + (\partial g^\vee)^{-1})^{-1}(0) = (\partial f \square \partial g^\vee)(0)$ . (iii): Using Theorem 4.7(i), Fermat's rule (see, e.g., [7, Theorem 16.2]) and (25) we have  $Z = (\text{zer } A) \cap (\text{zer } B) = \text{Argmin } f \cap \text{Argmin } g = (\partial f)^{-1}(0) \cap (\partial g)^{-1}(0) = \partial f^*(0) \cap \partial g^*(0)$ . (iv): Combine (i) and [7, Proposition 24.27] applied to the functions  $f^*$  and  $g^*$ . (v): Combine (ii) and [7, Proposition 24.27] applied to the functions  $f$  and  $g^\vee$ . ■

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<sup>9</sup>Let  $f : X \rightarrow ]-\infty, +\infty]$  be proper, convex and lower semicontinuous. We use  $f^*$  to denote the *convex conjugate* (a.k.a. Fenchel conjugate) of  $f$ , defined by  $f^* : X \rightarrow ]-\infty, +\infty] : x \mapsto \sup_{u \in X} (\langle x, u \rangle - f(u))$ .

<sup>10</sup>Let  $f : X \rightarrow ]-\infty, +\infty]$ . Then  $f^\vee : X \rightarrow ]-\infty, +\infty] : x \mapsto f(-x)$ .

<sup>11</sup>Let  $f : X \rightarrow ]-\infty, +\infty]$  be proper. The set of minimizers of  $f$ ,  $\{x \in X \mid f(x) = \inf f(X)\}$ , is denoted by  $\text{Argmin } f$ .

<sup>12</sup>Let  $S$  be nonempty subset of  $X$ . The *strong relative interior* of  $S$ , denoted by  $\text{sri } S$ , is the interior with respect to the closed affine hull of  $S$ .

<sup>13</sup>Let  $f : X \rightarrow ]-\infty, +\infty]$  and  $g : X \rightarrow ]-\infty, +\infty]$  be proper, convex and lower semicontinuous. The *infimal convolution* of  $f$  and  $g$ , denoted by  $f \square g$ , is the convex function  $f \square g : X \rightarrow \mathbb{R} : x \mapsto \inf_{y \in X} (f(y) + g(x - y))$ .

## 5 The Douglas-Rachford algorithm in the affine case

In this section we assume<sup>14</sup> that

$$A : X \rightrightarrows X \text{ and } B : X \rightrightarrows X \text{ are maximally monotone and affine,}$$

and that

$$Z = \{x \in X \mid 0 \in Ax + Bx\} \neq \emptyset. \quad (26)$$

Since the resolvents  $J_A$  and  $J_B$  are affine (see [9, Theorem 2.1(xix)]), so is  $T_{\text{DR}}$ .

**Theorem 5.1.** *Let  $x \in X$ . Then the following hold:*

- (i)  $T_{\text{DR}}^n x \rightarrow P_{\text{Fix } T_{\text{DR}}} x$ .
- (ii) Suppose that  $A$  and  $B$  are paramonotone such that  $K \perp (Z - Z)$  (as is the case when<sup>15</sup>  $A$  and  $B$  are paramonotone and  $(\text{zer } A) \cap (\text{zer } B) \neq \emptyset$ ). Then  $J_A T_{\text{DR}}^n x \rightarrow P_Z x$ .
- (iii) Suppose that  $X$  is finite-dimensional. Then  $T_{\text{DR}}^n x \rightarrow P_{\text{Fix } T_{\text{DR}}} x$  linearly and  $J_A T_{\text{DR}}^n x \rightarrow J_A P_{\text{Fix } T_{\text{DR}}} x$  linearly.

*Proof.* (i): Note that in view of Fact 4.3(ii) and (26) we have  $\text{Fix } T_{\text{DR}} \neq \emptyset$ . Moreover Fact 4.3(i) and Fact 2.1 imply that  $T_{\text{DR}}$  is asymptotically regular. It follows from Theorem 3.3 that (i) holds. (ii): Use (i) and Theorem 4.5. (iii): The linear convergence of  $(T_{\text{DR}}^n x)_{n \in \mathbb{N}}$  follows from Corollary 3.6. The linear convergence of  $(J_A T_{\text{DR}}^n x)_{n \in \mathbb{N}}$  is a direct consequence of the linear convergence of  $(T_{\text{DR}}^n x)_{n \in \mathbb{N}}$  and the fact that  $J_A$  is (firmly) nonexpansive. ■

**Remark 5.2.** *Theorem 5.1 generalizes the convergence results for the original Douglas-Rachford algorithm [21] from particular symmetric matrices/affine operators on a finite-dimensional space to general affine relations defined on possibly infinite dimensional spaces, while keeping strong and linear convergence of the iterates of the governing sequence  $(T_{\text{DR}}^n x)_{n \in \mathbb{N}}$  and identifying the limit to be  $P_{\text{Fix } T_{\text{DR}}} x$ . Paramonotonicity coupled with common zeros yields convergence of the shadow sequence  $(J_A T_{\text{DR}}^n x)_{n \in \mathbb{N}}$  to  $P_Z x$ .*

Suppose that  $U$  and  $V$  are nonempty closed convex subsets of  $X$ . Then

$$T_{U,V} = T_{\text{DR}}(N_U, N_V) = \text{Id} - P_U + P_V(2P_U - \text{Id}). \quad (27)$$

**Proposition 5.3.** *Suppose that  $U$  and  $V$  are closed linear subspaces of  $X$ . Let  $w \in X$ . Then  $w + U$  and  $w + V$  are closed affine subspaces of  $X$ ,  $(w + U) \cap (w + V) \neq \emptyset$  and  $(\forall n \in \mathbb{N})$*

$$T_{w+U, w+V}^n = T_{U,V}^n(\cdot - w) + w. \quad (28)$$

<sup>14</sup>  $A : X \rightrightarrows X$  is an *affine relation* if  $\text{gra } A$  is affine subspace of  $X \times X$ , i.e., a translation of a linear subspace of  $X \times X$ . For further information of affine relations we refer the reader to [12].

<sup>15</sup> See Theorem 4.7(iii).

*Proof.* Let  $x \in X$ . We proceed by induction. The case  $n = 0$  is clear. We now prove the case when  $n = 1$ , i.e.,

$$T_{w+U,w+V} = T_{U,V}(\cdot - w) + w. \quad (29)$$

Indeed,  $T_{w+U,w+V}x = (\text{Id} - P_{w+U} + P_{w+V}(2P_{w+U} - \text{Id}))x = x - w - P_U(x - w) + w + P_V(2P_{w+U}x - x - w) = x - w - P_U(x - w) + w + P_V(2w + 2P_U(x - w) - x - w) = (x - w) - P_U(x - w) + P_V(2P_U(x - w) - (x - w)) + w = (\text{Id} - P_U + P_V R_U)(x - w) + w = T_{V,U}(x - w) + w$ . We now assume that (28) holds for some  $n \in \mathbb{N}$ . Applying (29) with  $x$  replaced by  $T_{w+V,w+U}^n x$  yields

$$\begin{aligned} T_{w+U,w+V}^{n+1}x &= T_{w+U,w+V}(T_{w+U,w+V}^n x) = T_{U,V}(T_{w+U,w+V}^n x - w) + w \\ &= T_{U,V}(T_{U,V}^n(x - w) + w - w) + w = T_{U,V}^{n+1}(x - w) + w; \end{aligned} \quad (30)$$

hence (28) holds for all  $n \in \mathbb{N}$ . ■

**Example 5.4 (Douglas-Rachford in the affine feasibility case ).** (see also [4, Corollary 4.5]) Suppose that  $U$  and  $V$  are closed linear subspaces of  $X$ . Let  $w \in X$  and let  $x \in X$ . Suppose that  $A = N_{w+U}$  and that  $B = N_{w+V}$ . Then  $T_{w+U,w+V}x = Lx + b$ , where  $L = T_{U,V}$  and  $b = w - T_{U,V}w$ . Moreover,

$$T_{w+V,w+U}^n x \rightarrow P_{\text{Fix } T_{w+V,w+U}} x \quad (31)$$

and

$$J_A T_{w+V,w+U}^n x = P_{w+U} T_{w+V,w+U}^n x \rightarrow P_Z x = P_{(w+V) \cap (w+U)} x. \quad (32)$$

Finally, if  $U + V$  is closed (as is the case when  $X$  is finite-dimensional) then the convergence is linear with rate  $c_F(U, V) < 1$ , where  $c_F(U, V)$  is the cosine of the Friedrich's angle<sup>16</sup> between  $U$  and  $V$ .

*Proof.* Using (28) with  $n = 1$  and the linearity of  $T_{U,V}$  we have

$$T_{w+U,w+V} = T_{U,V}(\cdot - w) + w = T_{U,V} + w - T_{U,V}w. \quad (33)$$

Hence  $L = T_{U,V}$  and  $b = w - T_{U,V}w$ , as claimed. To obtain (31) and (32), use Theorem 5.1(i) and Theorem 5.1(ii), respectively. The claim about the linear rate follows by combining [4, Corollary 4.4] and Theorem 3.5 with  $T$  replaced by  $T_{w+U,w+V}$ ,  $L$  replaced by  $T_{U,V}$  and  $b$  replaced by  $w - T_{U,V}w$ . ■

**Remark 5.5.** When  $X$  is infinite-dimensional, it is possible to construct an example (see [4, Section 6]) of two linear subspaces  $U$  and  $V$  where  $c_F(U, V) = 1$ , and the rate of convergence of  $T_{\text{DR}}$  is not linear.

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<sup>16</sup> Suppose that  $U$  and  $V$  are closed linear subspaces of  $X$ . The cosine of the Friedrichs angle is  $c_F(\text{par } U, \text{par } V) = \sup_{\substack{u \in \text{par } U \cap W^\perp \cap \text{ball}(0;1) \\ v \in \text{par } V \cap W^\perp \cap \text{ball}(0;1)}} |\langle u, v \rangle| < 1$ , where  $W = \text{par } U \cap \text{par } V$ .

The assumption that both operators be paramonotone is critical for the conclusion in Theorem 5.1(ii), as shown below.

**Example 5.6.** Suppose that  $X = \mathbb{R}^2$ , that

$$A = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \text{ and that } B = N_{\{0\} \times \mathbb{R}}. \quad (34)$$

Then  $T_{\text{DR}} : \mathbb{R}^2 \rightarrow \mathbb{R}^2 : (x, y) \mapsto \frac{1}{2}(x - y) \cdot (1, -1)$ ,  $\text{Fix } T_{\text{DR}} = \mathbb{R} \cdot (1, -1)$ ,  $Z = \{0\} \times \mathbb{R}$ ,  $K = \{0\}$ , hence  $K \perp (Z - Z)$ , and  $(\forall (x, y) \in \mathbb{R}^2)(\forall n \geq 1) T_{\text{DR}}^n(x, y) = T_{\text{DR}}(x, y) = \frac{1}{2}(x - y, y - x) \in \text{Fix } T$ , however  $(\forall (x, y) \in (\mathbb{R} \setminus \{0\}) \times \mathbb{R})(\forall n \geq 1)$

$$(0, y - x) = J_A T_{\text{DR}}^n(x, y) \neq P_Z(x, y) = (0, y). \quad (35)$$

Note that  $A$  is not paramonotone by Example 4.2.

*Proof.* We have

$$J_A = (\text{Id} + A)^{-1} = \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}^{-1} = \frac{1}{2} \begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix}, \quad (36)$$

and

$$R_A = 2J_A - \text{Id} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}. \quad (37)$$

Moreover, by [7, Example 23.4],

$$J_B = P_{\mathbb{R} \times \{0\}} = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}. \quad (38)$$

Consequently

$$T_{\text{DR}} = \text{Id} - J_A + J_B R_A = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} - \frac{1}{2} \begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} = \frac{1}{2} \begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix}, \quad (39)$$

i.e.,

$$T_{\text{DR}} : \mathbb{R}^2 \rightarrow \mathbb{R}^2 : (x, y) \mapsto \frac{x-y}{2}(1, -1). \quad (40)$$

Now let  $(x, y) \in \mathbb{R}^2$ . Then  $(x, y) \in \text{Fix } T_{\text{DR}} \Leftrightarrow (x, y) = (\frac{x-y}{2}, -\frac{x-y}{2}) \Leftrightarrow x = \frac{x-y}{2}$  and  $y = -\frac{x-y}{2} \Leftrightarrow x + y = 0$ , hence  $\text{Fix } T_{\text{DR}} = \mathbb{R} \cdot (1, -1)$  as claimed. It follows from [19, Lemma 2.6(iii)] that  $Z = J_A(\text{Fix } T_{\text{DR}}) = \mathbb{R} \cdot J_A(1, -1) = \mathbb{R} \cdot \frac{1}{2}(0, 2) = \{0\} \times \mathbb{R}$ , as claimed. Now let  $(x, y) \in \mathbb{R}^2$ . By (40) we have  $T_{\text{DR}}(x, y) = \frac{x-y}{2}(1, -1) \in \text{Fix } T_{\text{DR}}$ , hence  $(\forall n \geq 1) T_{\text{DR}}^n(x, y) = T(x, y) = \frac{x-y}{2}(1, -1)$ . Therefore,  $(\forall n \geq 1) J_A T_{\text{DR}}^n(x, y) = J_A T_{\text{DR}}(x, y) = J_A \left( \frac{x-y}{2}(1, -1) \right) = (0, y - x) \neq (0, y) = P_Z(x, y)$  whenever  $x \neq 0$ .  $\blacksquare$

The next example illustrates that the assumption  $K \perp (Z - Z)$  is critical for the conclusion in Theorem 5.1(ii).

**Example 5.7 (when  $K \not\subseteq (Z - Z)$ ).** Let  $u \in X \setminus \{0\}$ . Suppose that  $A : X \rightarrow X : x \mapsto u$  and  $B : X \rightarrow X : x \mapsto -u$ . Then  $A$  and  $B$  are paramonotone,  $A + B \equiv 0$  and therefore  $Z = X$ . Moreover, by [6, Remark 5.4]  $(\forall z \in Z = X) K = (Az) \cap (-Bz) = \{u\} \not\subseteq (Z - Z) = X$ . Note that  $\text{Fix } T = Z + K = X + \{u\} = X$  and  $J_A : X \rightarrow X : x \mapsto x - u$ . Consequently

$$(\forall x \in X)(\forall n \in \mathbb{N}) \quad J_A T_{\text{DR}}^n x = J_A P_{\text{Fix } T} x = J_A x = x - u \neq x = P_Z x. \quad (41)$$

**Proposition 5.8 (parallel splitting).** Let  $m \in \{2, 3, \dots\}$ , and let  $B_i : X \rightrightarrows X$  be maximally monotone and affine,  $i \in \{1, 2, \dots, m\}$ , such that  $\text{zer}(\sum_{i=1}^m B_i) \neq \emptyset$ . Set  $\Delta = \{(x, \dots, x) \in X^m \mid x \in X\}$ , set  $\mathbf{A} = N_\Delta$ , set  $\mathbf{B} = \times_{i=1}^m B_i$ , set  $\mathbf{T} = T_{\text{DR}}(\mathbf{A}, \mathbf{B})$ , let  $j : X \rightarrow X^m : x \mapsto (x, x, \dots, x)$ , and let  $e : X^m \rightarrow X : (x_1, x_2, \dots, x_m) \mapsto \frac{1}{m} (\sum_{i=1}^m x_i)$ . Let  $\mathbf{x} \in X^m$ . Then  $\Delta^\perp = \{(u_1, \dots, u_m) \in X^m \mid \sum_{i=1}^m u_i = 0\}$ ,

$$\mathbf{Z} = Z_{(\mathbf{A}, \mathbf{B})} = j(\text{zer}(\sum_{i=1}^m B_i)) \subseteq \Delta \quad \text{and} \quad \mathbf{K} = K_{(\mathbf{A}, \mathbf{B})} = (-\mathbf{B}(\mathbf{Z})) \cap \Delta^\perp \subseteq \Delta^\perp. \quad (42)$$

Moreover, the following hold:

- (i)  $\mathbf{T}^n \mathbf{x} \rightarrow P_{\text{Fix } \mathbf{T}} \mathbf{x}$ .
- (ii) Suppose that  $X$  is finite-dimensional. Then  $\mathbf{T}^n \mathbf{x} \rightarrow P_{\text{Fix } \mathbf{T}} \mathbf{x}$  linearly and  $J_{\mathbf{A}} \mathbf{T}^n \mathbf{x} = P_\Delta \mathbf{T}^n \mathbf{x} \rightarrow P_\Delta P_{\text{Fix } \mathbf{T}} \mathbf{x}$  linearly.
- (iii) Suppose that  $B_i : X \rightrightarrows X$ ,  $i \in \{1, 2, \dots, m\}$ , are paramonotone. Then  $\mathbf{B}$  is paramonotone and  $J_{\mathbf{A}} \mathbf{T}^n \mathbf{x} = P_\Delta \mathbf{T}^n \mathbf{x} \rightarrow P_{\mathbf{Z}} \mathbf{x}$ . Consequently,  $e(J_{\mathbf{A}} \mathbf{T}^n \mathbf{x}) = e(P_\Delta \mathbf{T}^n \mathbf{x}) \rightarrow e(P_{\mathbf{Z}} \mathbf{x}) \in Z$ .

*Proof.* The claim about  $\Delta^\perp$  and first identity in (42) follows from [7, Proposition 25.5(i)&(vi)], whereas the second identity in (42) follows from Corollary 4.6(iii) applied to  $(\mathbf{A}, \mathbf{B})$ . (i): Apply Theorem 5.1(i) to  $(\mathbf{A}, \mathbf{B})$ . (ii): Apply Theorem 5.1(iii) to  $(\mathbf{A}, \mathbf{B})$ . (iii): Let  $(\mathbf{x}, \mathbf{u}), (\mathbf{y}, \mathbf{v})$  be in  $\text{gra } \mathbf{B}$ . On the one hand  $\langle \mathbf{x} - \mathbf{y}, \mathbf{u} - \mathbf{v} \rangle = 0 \Leftrightarrow \sum_{i=1}^m \langle x_i - y_i, u_i - v_i \rangle = 0$ ,  $(x_i, u_i), (y_i, v_i)$  are in  $\text{gra } B_i$ ,  $i \in \{1, \dots, m\}$ . On the other hand, since  $(\forall i \in \{1, \dots, m\}) B_i$  are monotone we learn that  $(\forall i \in \{1, \dots, m\}) \langle x_i - y_i, u_i - v_i \rangle \geq 0$ . Altogether,  $(\forall i \in \{1, \dots, m\}) \langle x_i - y_i, u_i - v_i \rangle = 0$ . Now use that paramonotonicity of  $B_i$  to deduce that  $(x_i, v_i), (y_i, u_i)$  are in  $\text{gra } B_i$ ,  $i \in \{1, \dots, m\}$ ; equivalently,  $(\mathbf{x}, \mathbf{v}), (\mathbf{y}, \mathbf{u})$  in  $\text{gra } \mathbf{B}$ . Finally, apply Corollary 4.6(iv).  $\blacksquare$

## 6 Examples of linear monotone operators

In this section we present examples of monotone operators that are partly motivated by applications in partial differential equations; see, e.g., [25] and [37]. Let  $M \in \mathbb{R}^{n \times n}$ . Then we have the following equivalences:

$$M \text{ is monotone} \Leftrightarrow \frac{M + M^\top}{2} \text{ is positive semidefinite} \quad (43a)$$

$$\Leftrightarrow \text{the eigenvalues of } \frac{M + M^T}{2} \text{ lie in } \mathbb{R}_+. \quad (43b)$$

**Lemma 6.1.** *Let*

$$M = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \in \mathbb{R}^{2 \times 2}. \quad (44)$$

*Then  $M$  is monotone if and only if  $\alpha \geq 0$ ,  $\delta \geq 0$  and  $4\alpha\delta \geq (\beta + \gamma)^2$ .*

*Proof.* Indeed, the principal minors of  $M + M^T$  are  $2\alpha$ ,  $2\delta$  and  $4\alpha\delta - (\beta + \gamma)^2$ ; by, e.g., [30, (7.6.12) on page 566]. ■

Note that if  $M = M^T$ , then  $M$  is monotone if and only if the eigenvalues of  $M$  lie in  $\mathbb{R}_+$ . If  $M \neq M^T$ , then some information about the location of the (possibly complex) eigenvalues of  $M$  is available:

**Lemma 6.2.** *Let  $M \in \mathbb{R}^{n \times n}$  be monotone, and let  $\{\lambda_k\}_{k=1}^n$  denote the set of eigenvalues of  $M$ . Then<sup>17</sup>  $\text{Re}(\lambda_k) \geq 0$  for every  $k \in \{1, \dots, n\}$ .*

*Proof.* Write  $\lambda = \alpha + i\beta$ , where  $\alpha$  and  $\beta$  belong to  $\mathbb{R}$  and  $i = \sqrt{-1}$  and assume that  $\lambda$  is an eigenvalue of  $M$  with (nonzero) eigenvector  $w = u + iv$ , where  $u$  and  $v$  are in  $\mathbb{R}^n$ . Then  $(M - \lambda \text{Id})w = 0 \Leftrightarrow ((M - \alpha \text{Id}) - i\beta \text{Id})(u + iv) = 0 \Leftrightarrow (M - \alpha \text{Id})u + \beta v = 0$  and  $(M - \alpha \text{Id})v - \beta u = 0$ . Hence

$$\langle u, (M - \alpha \text{Id})u \rangle + \beta \langle u, v \rangle = 0, \quad (45a)$$

$$\langle v, (M - \alpha \text{Id})v \rangle - \beta \langle v, u \rangle = 0. \quad (45b)$$

Adding (45) yields  $\langle u, (M - \alpha \text{Id})u \rangle + \langle v, (M - \alpha \text{Id})v \rangle = 0$ ; equivalently,  $\langle u, Mu \rangle + \langle v, Mv \rangle - \alpha \|u\|^2 - \alpha \|v\|^2 = 0$ . Solving for  $\alpha$  yields

$$\text{Re}(\lambda) = \alpha = \frac{\langle u, Mu \rangle + \langle v, Mv \rangle}{\|u\|^2 + \|v\|^2} \geq 0, \quad (46)$$

as claimed. ■

The converse of Lemma 6.2 is not true in general, as we demonstrate in the following example.

**Example 6.3.** *Let  $\xi \in \mathbb{R} \setminus [-2, 2]$ , and set*

$$M = \begin{pmatrix} 1 & \xi \\ 0 & 1 \end{pmatrix}. \quad (47)$$

*Then  $M$  has 1 as its only eigenvalue (with multiplicity 2),  $M$  is not monotone by Lemma 6.1, and  $M$  is not symmetric.*

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<sup>17</sup>Let  $\mathbb{C}$  be the set of complex numbers and let  $z \in \mathbb{C}$ . We use  $\text{Re}(z)$  to refer to the real part of the complex number  $z$ .



**Proposition 6.4.** Consider the tridiagonal Toeplitz matrix

$$M = \begin{pmatrix} \beta & \gamma & & 0 \\ \alpha & \ddots & \ddots & \\ & \ddots & \ddots & \gamma \\ 0 & & \alpha & \beta \end{pmatrix} \in \mathbb{R}^{n \times n}. \quad (48)$$

Then  $M$  is monotone if and only if  $\beta \geq |\alpha + \gamma| \cos(\pi/(n+1))$ .

*Proof.* Note that

$$\frac{1}{2}(M + M^T) = \begin{pmatrix} \beta & \frac{1}{2}(\alpha + \gamma) & & 0 \\ \frac{1}{2}(\alpha + \gamma) & \ddots & \ddots & \\ & \ddots & \ddots & \frac{1}{2}(\alpha + \gamma) \\ 0 & & \frac{1}{2}(\alpha + \gamma) & \beta \end{pmatrix}. \quad (49)$$

By (43a),  $M$  is monotone  $\Leftrightarrow \frac{1}{2}(M + M^T)$  is positive semidefinite. If  $\alpha + \gamma = 0$  then  $\frac{1}{2}(M + M^T) = \beta \text{Id}$  and therefore  $\frac{1}{2}(M + M^T)$  is positive semidefinite  $\Leftrightarrow \beta \geq 0 = |\alpha + \gamma|$ . Now suppose that  $\alpha + \gamma \neq 0$ . It follows from [30, Example 7.2.5] that the eigenvalues of  $\frac{1}{2}(M + M^T)$  are

$$\lambda_k = \beta + (\alpha + \gamma) \cos\left(\frac{k\pi}{n+1}\right), \quad (50)$$

where  $k \in \{1, \dots, n\}$ . Consequently,  $\{\lambda_k\}_{k=1}^n \subseteq \mathbb{R}_+ \Leftrightarrow \beta \geq |(\alpha + \gamma) \cos(\pi/(n+1))|$ . Therefore, the characterization of monotonicity of  $M$  follows from (43b). ■

**Proposition 6.5.** Let

$$M = \begin{pmatrix} \beta & \gamma & & 0 \\ \alpha & \ddots & \ddots & \\ & \ddots & \ddots & \gamma \\ 0 & & \alpha & \beta \end{pmatrix} \in \mathbb{R}^{n \times n}. \quad (51)$$

Then exactly one of the following holds:

(i)  $\alpha\gamma = 0$  and  $\det(M) = \beta^n$ . Consequently  $M$  is invertible  $\Leftrightarrow \beta \neq 0$ , in which case

$$[M^{-1}]_{i,j} = (-\alpha)^{\max\{i-j, 0\}} (-\gamma)^{\max\{j-i, 0\}} \beta^{\min\{j-i, i-j\}-1}. \quad (52)$$

(ii)  $\alpha\gamma \neq 0$ . Set  $r = \frac{1}{2\alpha}(-\beta + \sqrt{\beta^2 - 4\alpha\gamma})$ ,  $s = \frac{1}{2\alpha}(-\beta - \sqrt{\beta^2 - 4\alpha\gamma})$  and  $\Lambda = \{\beta + 2\gamma\sqrt{\alpha/\gamma} \cos(k\pi/(n+1)) \mid k \in \{1, 2, \dots, n\}\}$ . Then  $rs \neq 0$ . Moreover,  $M$  is invertible<sup>18</sup>  $\Leftrightarrow 0 \notin \Lambda$ , in which case

$$r \neq s \Rightarrow [M^{-1}]_{i,j} = -\frac{\gamma^{j-1}(r^{\min\{i,j\}} - s^{\min\{i,j\}})(r^{n+1}s^{\max\{i,j\}} - r^{\max\{i,j\}}s^{n+1})}{\alpha^j(r-s)(r^{n+1} - s^{n+1})}, \quad (53a)$$

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<sup>18</sup>In the special case, when  $\beta = 0$ , this is equivalent to saying that  $M$  is invertible  $\Leftrightarrow n$  is even.

$$r = s \Rightarrow [M^{-1}]_{i,j} = -\frac{\gamma^{j-1} \min\{i, j\} (n+1 - \max\{i, j\}) r^{i+j-1}}{\alpha^j (n+1)}. \quad (53b)$$

Alternatively, define the recurrence relations

$$u_0 = 0, \quad u_1 = 1, \quad u_k = -\frac{1}{\gamma}(\alpha u_{k-2} + \beta u_{k-1}), \quad k \geq 2; \quad (54a)$$

$$v_{n+1} = 0, \quad v_n = 1, \quad v_k = -\frac{1}{\alpha}(\beta v_{k+1} + \gamma v_{k+2}), \quad k \leq n-1. \quad (54b)$$

Then

$$[M^{-1}]_{i,j} = -\frac{u_{\min\{i,j\}} v_{\max\{i,j\}}}{v_0} \left(\frac{\gamma}{\alpha}\right)^{j-1}. \quad (55)$$

*Proof.* (i):  $\alpha\gamma = 0 \Leftrightarrow \alpha = 0$  or  $\gamma = 0$ , in which case  $M$  is a (lower or upper) triangular matrix. Hence  $\det(M) = \beta^n$ , and the characterization follows. The formula in (52) is easily verified. (ii): Note that  $0 \in \{r, s\} \Leftrightarrow \beta \in \{\pm\sqrt{\beta^2 - 4\alpha\gamma}\} \Leftrightarrow \beta^2 = \beta^2 - 4\alpha\gamma \Leftrightarrow \alpha\gamma = 0$ . Hence  $rs \neq 0$ . Moreover, it follows from [30, Example 7.2.5] that  $\Lambda$  is the set of eigenvalues of  $M$ ; therefore,  $M$  is invertible  $\Leftrightarrow 0 \notin \Lambda$ . The formulae (53) follow from [38, Remark 2 on page 110]. The recurrence formulae defined in (54) and [38, Theorem 2] yield (55). ■

**Remark 6.6.** Concerning Proposition 6.5, it follows from [36, Section 2 on page 44] that we also have the alternative formulae

$$r \neq s \Rightarrow [M^{-1}]_{i,j} = \begin{cases} -\frac{1}{\gamma} \frac{s^{-i} - r^{-i}}{s^{-1} - r^{-1}} \frac{s^{-n+j-1} - r^{-n+j-1}}{s^{-(n+1)} - r^{-(n+1)}}, & j \geq i; \\ -\frac{1}{\alpha} \frac{s^j - r^j}{s - r} \frac{s^{n-i+1} - r^{n-i+1}}{s^{n+1} - r^{n+1}}, & j \leq i, \end{cases} \quad (56a)$$

$$r = s \Rightarrow [M^{-1}]_{i,j} = \begin{cases} -\frac{i}{\gamma} \left(1 - \frac{j}{n+1}\right) r^{j-i+1}, & j \geq i; \\ -\frac{j}{\alpha} \left(1 - \frac{i}{n+1}\right) r^{j-i-1}, & j \leq i. \end{cases} \quad (56b)$$

Using the binomial expansion, (56a), and a somewhat tedious calculation which we omit here, one can show that  $[M^{-1}]_{i,j}$  is equal to

$$- \frac{2 \left( \sum_{m=0}^{\lceil \min\{i,j\}/2 \rceil - 1} \binom{\min\{i,j\}}{2m+1} (-\beta)^{\min\{i,j\} - (2m+1)} (\beta^2 - 4\alpha\gamma)^m \right) \left( \sum_{m=0}^{\lceil (n+1 - \max\{i,j\})/2 \rceil - 1} \binom{n - \max\{i,j\} + 1}{2m+1} (-\beta)^{n - \max\{i,j\} - 2m} (\beta^2 - 4\alpha\gamma)^m \right)}{(2\alpha)^{\min\{0, j-i\}} (2\gamma)^{\min\{0, i-j\}} \left( \sum_{m=0}^{\lceil (n+1)/2 \rceil - 1} \binom{n+1}{2m+1} (-\beta)^{n-2m} (\beta^2 - 4\alpha\gamma)^m \right)} \quad (57)$$

provided that  $r \neq s$ .

**Example 6.7.** Let  $\beta \geq 2$ , set

$$M = \begin{pmatrix} \beta & -1 & & 0 \\ -1 & \ddots & \ddots & \\ & \ddots & \ddots & -1 \\ 0 & & -1 & \beta \end{pmatrix} \in \mathbb{R}^{n \times n}, \quad (58)$$

set  $r = \frac{1}{2}(\beta + \sqrt{\beta^2 - 4})$  and set  $s = \frac{1}{2}(\beta - \sqrt{\beta^2 - 4})$ . Then  $M$  is monotone and invertible. Moreover,

$$r \neq s \Rightarrow [M^{-1}]_{i,j} = \frac{(r^{\min\{i,j\}} - s^{\min\{i,j\}})(r^{n+1} s^{\max\{i,j\}} - r^{\max\{i,j\}} s^{n+1})}{(r - s)(r^{n+1} - s^{n+1})}, \quad (59a)$$

$$r = s \Rightarrow [M^{-1}]_{i,j} = \frac{\min\{i,j\} (n+1 - \max\{i,j\})}{n+1}. \quad (59b)$$

Alternatively, define the recurrence relations

$$u_0 = 0, \quad u_1 = 1, \quad u_k = \beta u_{k-1} - u_{k-2}, \quad k \geq 2, \quad (60a)$$

$$v_{n+1} = 0, \quad v_n = 1, \quad v_k = \beta v_{k+1} - v_{k+2}, \quad k \leq n-1. \quad (60b)$$

Then

$$[M^{-1}]_{i,j} = -\frac{u_{\min\{i,j\}} v_{\max\{i,j\}}}{v_0}. \quad (61)$$

*Proof.* The monotonicity of  $M$  follows from Proposition 6.4 by noting that  $\beta \geq 2 > 2 \cos(\pi/(n+1))$ . The same argument implies that

$$0 \notin \Lambda = \left\{ \beta - 2 \cos\left(\frac{k\pi}{n+1}\right) \mid k \in \{1, 2, \dots, n\} \right\}. \quad (62)$$

Hence  $M$  is invertible by Proposition 6.5(ii). Note that  $\beta = 2 \Leftrightarrow \beta^2 - 4 = 0 \Leftrightarrow r = s = 1$ . Now apply Proposition 6.5(ii).  $\blacksquare$

Let  $M_1 = [\alpha_{i,j}]_{i,j=1}^n \in \mathbb{R}^{n \times n}$  and  $M_2 = [\beta_{i,j}]_{i,j=1}^n \in \mathbb{R}^{n \times n}$ . Recall that the *Kronecker product* of  $M_1$  and  $M_2$  (see, e.g., [28, page 407] or [30, Exercise 7.6.10]) is defined by the block matrix

$$M_1 \otimes M_2 = [\alpha_{i,j} M_2] \in \mathbb{R}^{n^2 \times n^2}. \quad (63)$$

**Lemma 6.8.** Let  $M_1$  and  $M_2$  be symmetric matrices in  $\mathbb{R}^{n \times n}$ . Then  $M_1 \otimes M_2 \in \mathbb{R}^{n^2 \times n^2}$  is symmetric.

*Proof.* Using [30, Exercise 7.8.11(a)] or [28, Proposition 1(e) on page 408] we have  $(M_1 \otimes M_2)^\top = M_1^\top \otimes M_2^\top = M_1 \otimes M_2$ .  $\blacksquare$

The following fact is very useful in the conclusion of the upcoming results.

**Fact 6.9.** Let  $M_1$  and  $M_2$  be in  $\mathbb{R}^{n \times n}$ , with eigenvalues  $\{\lambda_k \mid k \in \{1, \dots, n\}\}$  and  $\{\mu_k \mid k \in \{1, \dots, n\}\}$ . Then the eigenvalues of  $M_1 \otimes M_2$  are  $\{\lambda_j \mu_k \mid j, k \in \{1, \dots, n\}\}$ .

*Proof.* See [28, Corollary 1 on page 412] or [30, Exercise 7.8.11(b)]. ■

**Corollary 6.10.** Let  $M_1$  and  $M_2$  in  $\mathbb{R}^{n \times n}$  be monotone such that  $M_1$  or  $M_2$  is symmetric. Then  $M_1 \otimes M_2$  is monotone.

*Proof.* According to (43), it suffices to show that all the eigenvalues of  $M_1 \otimes M_2 + (M_1 \otimes M_2)^\top$  are nonnegative. Suppose first that  $M_1$  is symmetric. Then using [28, Proposition 1(e)&(c)] we have  $M_1 \otimes M_2 + (M_1 \otimes M_2)^\top = M_1 \otimes M_2 + M_1 \otimes M_2^\top = M_1 \otimes (M_2 + M_2^\top)$ . Since  $M_2$  is monotone, it follows from (43) that all the eigenvalues of  $M_2 + M_2^\top$  are nonnegative. Now apply Fact 6.9 to  $M_1$  and  $M_2 + M_2^\top$  to learn that all the eigenvalues of  $M_1 \otimes M_2 + (M_1 \otimes M_2)^\top$  are nonnegative, hence  $M_1 \otimes M_2$  is monotone by (43b). A similar argument applies if  $M_2$  is monotone. ■

Note that the assumption that at least one matrix is symmetric is critical in Corollary 6.10, as we show in the next example.

**Example 6.11.** Suppose that

$$M = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}. \quad (64)$$

Then  $M$  is monotone, with eigenvalues  $\{\pm i\}$ , but not symmetric. However,

$$M \otimes M = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \\ 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}. \quad (65)$$

is a symmetric matrix with eigenvalues  $\{\pm 1\}$  by Fact 6.9. Therefore  $M \otimes M$  is not monotone by (43).

**Proposition 6.12.** Let  $M \in \mathbb{R}^{n \times n}$  be symmetric. Then  $\text{Id} \otimes M$  is monotone  $\Leftrightarrow M \otimes \text{Id}$  is monotone  $\Leftrightarrow M$  is monotone, in which case we have

$$J_{\text{Id}_n \otimes M} = \text{Id}_n \otimes J_M \quad \text{and} \quad J_{M \otimes \text{Id}_n} = J_M \otimes \text{Id}_n. \quad (66)$$

*Proof.* In view of Fact 6.9 the sets of eigenvalues of  $\text{Id} \otimes M$ ,  $M \otimes \text{Id}$ , and  $M$  coincide. It follows from Lemma 6.8 that  $\text{Id} \otimes M$  and  $M \otimes \text{Id}$  are symmetric. Now apply (43b) and use the monotonicity of  $M$ . To prove (66), we use [28, Proposition 1(c) on page 408] to learn that  $\text{Id}_{n^2} + \text{Id}_n \otimes M = \text{Id}_n \otimes \text{Id}_n + \text{Id}_n \otimes M = \text{Id}_n \otimes (\text{Id}_n + M)$ . Therefore, by [28, Corollary 1(b) on page 408] we have

$$J_{\text{Id}_n \otimes M} = (\text{Id}_{n^2} + \text{Id}_n \otimes M)^{-1} = (\text{Id}_n \otimes (\text{Id}_n + M))^{-1} = \text{Id}_n \otimes (\text{Id}_n + M)^{-1}$$

$$= \text{Id}_n \otimes J_M. \quad (67)$$

The other identity in (66) is proved similarly.  $\blacksquare$

**Corollary 6.13.** *Let  $\beta \in \mathbb{R}$ . Set*

$$M_{[\beta]} = \begin{pmatrix} \beta & -1 & & 0 \\ -1 & \ddots & \ddots & \\ & \ddots & \ddots & -1 \\ 0 & & -1 & \beta \end{pmatrix} \in \mathbb{R}^{n \times n}, \quad (68)$$

and let  $\mathbf{M}_{\rightarrow}$  and  $\mathbf{M}_{\uparrow}$  be block matrices in  $\mathbb{R}^{n^2 \times n^2}$  defined by

$$\mathbf{M}_{\rightarrow} = \begin{pmatrix} M_{[\beta]} & 0_n & & 0_n \\ 0_n & \ddots & \ddots & \\ & \ddots & \ddots & 0_n \\ 0_n & & 0_n & M_{[\beta]} \end{pmatrix} \quad \text{and} \quad \mathbf{M}_{\uparrow} = \begin{pmatrix} \beta \text{Id}_n & -\text{Id}_n & & 0_n \\ -\text{Id}_n & \ddots & \ddots & \\ & \ddots & \ddots & -\text{Id}_n \\ 0_n & & -\text{Id}_n & \beta \text{Id}_n \end{pmatrix}. \quad (69)$$

Then  $\mathbf{M}_{\rightarrow} = \text{Id}_n \otimes M_{[\beta]}$  and  $\mathbf{M}_{\uparrow} = M_{[\beta]} \otimes \text{Id}_n$ . Moreover,  $\mathbf{M}_{\rightarrow}$  is monotone  $\Leftrightarrow \mathbf{M}_{\uparrow}$  is monotone  $\Leftrightarrow M_{[\beta]}$  is monotone  $\Leftrightarrow \beta \geq 2 \cos(\pi/(n+1))$ , in which case

$$J_{\mathbf{M}_{\rightarrow}} = \text{Id}_n \otimes M_{[\beta+1]}^{-1} \quad \text{and} \quad J_{\mathbf{M}_{\uparrow}} = M_{[\beta+1]}^{-1} \otimes \text{Id}_n. \quad (70)$$

*Proof.* It is straightforward to verify that  $\mathbf{M}_{\rightarrow} = \text{Id}_n \otimes M_{[\beta]}$  and  $\mathbf{M}_{\uparrow} = M_{[\beta]} \otimes \text{Id}_n$ . It follows from Proposition 6.4 that  $M_{[\beta]}$  is monotone  $\Leftrightarrow \beta \geq 2 \cos(\frac{\pi}{n+1})$ . Now combine with Proposition 6.12. To prove (70) note that  $\text{Id}_n + M_{[\beta]} = M_{[\beta+1]}$ , and therefore  $J_{M_{[\beta]}} = M_{[\beta+1]}^{-1}$ . The conclusion follows by applying (66).  $\blacksquare$

The above matrices play a key role in the original design of the Douglas-Rachford algorithm — see the Appendix for details.

**Proposition 6.14.** *Let  $n \in \{2, 3, \dots\}$ , let  $M \in \mathbb{R}^{n \times n}$  and consider the block matrix*

$$\mathbf{M} = \begin{pmatrix} M & -\text{Id}_n & & 0_n \\ -\text{Id}_n & \ddots & \ddots & \\ & \ddots & \ddots & -\text{Id}_n \\ 0_n & & -\text{Id}_n & M \end{pmatrix}. \quad (71)$$

Let  $x = (x_1, x_2, \dots, x_n) \in \mathbb{R}^{n^2}$ , where  $x_i \in \mathbb{R}^n, i \in \{1, 2, \dots, n\}$ . Then

$$\begin{aligned} \langle x, \mathbf{M}x \rangle &= \langle x_1, (M - \text{Id})x_1 \rangle + \sum_{k=2}^{n-1} \langle x_k, (M - 2\text{Id})x_k \rangle \\ &\quad + \langle x_n, (M - \text{Id})x_n \rangle + \sum_{i=1}^{n-1} \|x_i - x_{i+1}\|^2. \end{aligned} \quad (72)$$

Moreover the following hold:

- (i) Suppose that  $n = 2$ . Then  $M - \text{Id}$  is monotone  $\Leftrightarrow \mathbf{M}$  is monotone.
- (ii)  $M - 2\text{Id}$  is monotone  $\Rightarrow \mathbf{M}$  is monotone.
- (iii)  $\mathbf{M}$  is monotone  $\Rightarrow M - 2(1 - \frac{1}{n})\text{Id}$  is monotone  $\Rightarrow M$  is monotone.

*Proof.* We have

$$\begin{aligned}
\langle x, \mathbf{M}x \rangle &= \langle (x_1, x_2, \dots, x_n), (Mx_1 - x_2, -x_1 + Mx_2 - x_3, \dots, -x_{n-1} + Mx_n) \rangle \\
&= \langle x_1, Mx_1 \rangle - \langle x_1, x_2 \rangle - \langle x_1, x_2 \rangle + \langle x_2, Mx_2 \rangle - \langle x_2, x_3 \rangle \\
&\quad - \langle x_2, x_3 \rangle + \langle x_3, Mx_3 \rangle - \dots - \langle x_{n-1}, x_n \rangle + \langle x_n, Mx_n \rangle - \langle x_{n-1}, x_n \rangle \\
&= \langle x_1, Mx_1 \rangle - 2\langle x_1, x_2 \rangle + \langle x_2, Mx_2 \rangle - 2\langle x_2, x_3 \rangle \\
&\quad - \dots - 2\langle x_{n-1}, x_n \rangle + \langle x_n, Mx_n \rangle - 2\langle x_{n-1}, x_n \rangle \\
&= \langle x_1, Mx_1 \rangle + \|x_1 - x_2\|^2 - \|x_1\|^2 - \|x_2\|^2 + \langle x_2, Mx_2 \rangle + \|x_2 - x_3\|^2 \\
&\quad - \|x_2\|^2 - \|x_3\|^2 + \dots + \|x_{n-1} - x_n\|^2 - \|x_{n-1}\|^2 - \|x_n\|^2 + \langle x_n, Mx_n \rangle \\
&= \langle x_1, Mx_1 \rangle - \|x_1\|^2 + \langle x_2, Mx_2 \rangle - 2\|x_2\|^2 + \dots + \langle x_n, Mx_n \rangle - \|x_n\|^2 + \dots \\
&\quad + \|x_1 - x_2\|^2 + \|x_2 - x_3\|^2 + \dots + \|x_{n-1} - x_n\|^2 \\
&= \langle x_1, (M - \text{Id})x_1 \rangle + \left( \sum_{k=2}^{n-1} \langle x_k, (M - 2\text{Id})x_k \rangle \right) + \langle x_n, (M - \text{Id})x_n \rangle \\
&\quad + \sum_{i=1}^{n-1} \|x_i - x_{i+1}\|^2.
\end{aligned}$$

(i): “ $\Rightarrow$ ”: Apply (72) with  $n = 2$ . “ $\Leftarrow$ ”: Let  $y \in \mathbb{R}^2$ . Applying (72) to the point  $x = (y, y) \in \mathbb{R}^4$ , we get  $0 \leq \langle x, \mathbf{M}x \rangle = 2\langle y, (\text{Id} - M)y \rangle$ . (ii): This is clear from (72). (iii): Let  $y \in \mathbb{R}^n$ . Applying (72) to the point  $x = (y, y, \dots, y) \in \mathbb{R}^{n^2}$  yields  $0 \leq \langle x, \mathbf{M}x \rangle = 2\langle y, (M - \text{Id})y \rangle + (n - 2)\langle y, (M - 2\text{Id})y \rangle = \langle y, (nM - 2(n - 1)\text{Id})y \rangle$ . Therefore,  $M - 2(1 - \frac{1}{n})\text{Id}$  is monotone.  $\blacksquare$

The converse of Proposition 6.14(ii) is not true in general, as we illustrate now.

**Example 6.15.** Set

$$M = \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}, \tag{73}$$

and let  $\mathbf{M}$  be as defined in Proposition 6.14. Then one verifies easily that  $\mathbf{M}$  is monotone while  $M - 2\text{Id}$  is not.

We now show that the converse of the implications in Proposition 6.14(iii) are not true in general.

**Example 6.16.** Set  $n = 2$ , set  $M = \frac{1}{2} \text{Id} \in \mathbb{R}^{2 \times 2}$ , and let  $\mathbf{M}$  be as defined in Proposition 6.14.. Then  $M$  is monotone but  $M - 2(1 - \frac{1}{2}) \text{Id} = -\frac{1}{2} \text{Id}$  is not monotone, and

$$\mathbf{M} = \frac{1}{2} \begin{pmatrix} 1 & 0 & -2 & 0 \\ 0 & 1 & 0 & -2 \\ -2 & 0 & 1 & 0 \\ 0 & -2 & 0 & 1 \end{pmatrix}. \quad (74)$$

Note the  $\mathbf{M}$  is symmetric and has eigenvalues  $\{-1/2, 3/2\}$ , hence  $\mathbf{M}$  is not monotone by (43).

## Acknowledgments

HHB was partially supported by the Natural Sciences and Engineering Research Council of Canada and by the Canada Research Chair Program.

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## Appendix

In this section we briefly show the connection between the original Douglas-Rachford algorithm introduced in [21] (see also [22], [31] and [20] for variations of this method) to solve certain types of heat equations and the general algorithm introduced by Lions and Mercier in [29] (see also [19]).

Suppose that  $\Omega$  is a bounded square region in  $\mathbb{R}^2$ . Consider the Dirichlet problem for the Poisson equation: Given  $f$  and  $g$ , find  $u : \Omega \rightarrow \mathbb{R}$  such that

$$\Delta u = f \text{ on } \Omega \quad \text{and} \quad u = g \text{ on } \text{bdry } \Omega, \quad (75)$$

where  $\Delta = \nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}$  is the Laplace operator and  $\text{bdry } \Omega$  denotes the boundary of  $\Omega$ . Discretizing  $u$  followed by converting it into a “long vector”  $y$  (see [30, Example 7.6.2 & Problem 7.6.9]) we obtain the system of linear equations

$$L_{\rightarrow} y + L_{\uparrow} y = -b. \quad (76)$$

Here  $L_{\rightarrow}$  and  $L_{\uparrow}$  denote the horizontal (respectively vertical) positive definite discretization of the negative Laplacian over a square mesh with  $n^2$  points at equally spaced intervals (see, [30, Problem 7.6.10]). We have

$$L_{\rightarrow} = \text{Id} \otimes M \quad \text{and} \quad L_{\uparrow} = M \otimes \text{Id}, \quad (77)$$

where

$$M = \begin{pmatrix} 2 & -1 & & 0 \\ -1 & \ddots & \ddots & \\ & \ddots & \ddots & -1 \\ 0 & & -1 & 2 \end{pmatrix} \in \mathbb{R}^{n \times n}. \quad (78)$$

To see the connection to monotone operators, set  $A = L_{\rightarrow}$  and  $B : L_{\uparrow} + b : y \mapsto L_{\uparrow}y + b$ . Then  $A$  and  $B$  are affine and strictly monotone. The problem then reduces to

$$\text{find } y \in \mathbb{R}^{n^2} \text{ such that } Ay + By = 0, \quad (79)$$

and the algorithm proposed by Douglas and Rachford in [21] becomes

$$y_{n+1/2} + Ay_n + By_{n+1/2} - y_n = 0, \quad (80a)$$

$$y_{n+1} - y_{n+1/2} - Ay_n + Ay_{n+1} = 0. \quad (80b)$$

Consequently,

$$(80a) \Leftrightarrow (\text{Id} + B)(y_{n+1/2}) = (\text{Id} - A)y_n \Leftrightarrow y_{n+1/2} = J_B(\text{Id} - A)y_n, \quad (81a)$$

$$(80b) \Leftrightarrow (\text{Id} + A)y_{n+1} = Ay_n + y_{n+1/2} \Leftrightarrow y_{n+1} = J_A(Ay_n + y_{n+1/2}). \quad (81b)$$

Substituting (81a) into (81b) to eliminate  $y_{n+1/2}$  yields

$$y_{n+1} = J_A(Ay_n + J_B(\text{Id} - A)y_n). \quad (82)$$

To proceed further, we must show that

$$(\text{Id} - A)J_A = R_A \quad (83a)$$

$$AJ_A = \text{Id} - J_A. \quad (83b)$$

Indeed, note that  $\text{Id} - A = 2\text{Id} - (\text{Id} + A)$ , therefore multiplying by  $J_A = (\text{Id} + A)^{-1}$  from the right yields  $(\text{Id} - A)J_A = (2\text{Id} - (\text{Id} + A))J_A = 2J_A - \text{Id} = R_A$ . Hence  $J_A - AJ_A = J_A - (\text{Id} - J_A)$ ; equivalently,  $AJ_A = \text{Id} - J_A$ . Now consider the change of variable

$$(\forall n \in \mathbb{N}) \quad x_n = (\text{Id} + A)y_n, \quad (84)$$

which is equivalent to  $y_n = J_A x_n$ . Substituting (82) into (84), and using (83), yield

$$\begin{aligned} x_{n+1} &= (\text{Id} + A)y_{n+1} = (\text{Id} + A)J_A(Ay_n + J_B(\text{Id} - A)y_n) = Ay_n + J_B(\text{Id} - A)y_n \\ &= AJ_A x_n + J_B(\text{Id} - A)J_A x_n = x_n - J_A x_n + J_B R_A x_n = (\text{Id} - J_A + J_B R_A)x_n, \end{aligned} \quad (85)$$

which is the Douglas-Rachford update formula (18).

We point out that  $J_A = J_{L \rightarrow}$ , and using [7, Proposition 23.15(ii)] we have  $J_B = J_{L \uparrow + b} = J_{L \uparrow} - J_{L \uparrow} b$ . To calculate  $J_A$  and  $J_B$  apply Corollary 6.13 to get

$$J_A = \text{Id}_n \otimes J_M \quad \text{and} \quad J_B = J_M \otimes \text{Id}_n - (J_M \otimes \text{Id}_n)(b). \quad (86)$$

For instance, when  $n = 3$ , the above calculations yield

$$J_M = \begin{pmatrix} \frac{8}{21} & \frac{1}{7} & \frac{1}{21} \\ \frac{1}{7} & \frac{3}{7} & \frac{1}{7} \\ \frac{1}{21} & \frac{1}{7} & \frac{8}{21} \end{pmatrix}, \quad (87)$$

$$\text{Id}_3 \otimes J_M = \begin{pmatrix} \frac{8}{21} & \frac{1}{7} & \frac{1}{21} & 0 & 0 & 0 & 0 & 0 & 0 \\ \frac{1}{7} & \frac{3}{7} & \frac{1}{7} & 0 & 0 & 0 & 0 & 0 & 0 \\ \frac{1}{21} & \frac{1}{7} & \frac{8}{21} & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{8}{21} & \frac{1}{7} & \frac{1}{21} & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{1}{7} & \frac{3}{7} & \frac{1}{7} & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{1}{21} & \frac{1}{7} & \frac{8}{21} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & \frac{8}{21} & \frac{1}{7} & \frac{1}{21} \\ 0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{7} & \frac{3}{7} & \frac{1}{7} \\ 0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{21} & \frac{1}{7} & \frac{8}{21} \end{pmatrix}, \quad (88)$$

and

$$J_M \otimes \text{Id}_3 = \begin{pmatrix} \frac{8}{21} & 0 & 0 & \frac{1}{7} & 0 & 0 & \frac{1}{21} & 0 & 0 \\ 0 & \frac{8}{21} & 0 & 0 & \frac{1}{7} & 0 & 0 & \frac{1}{21} & 0 \\ 0 & 0 & \frac{8}{21} & 0 & 0 & \frac{1}{7} & 0 & 0 & \frac{1}{21} \\ \frac{1}{7} & 0 & 0 & \frac{3}{7} & 0 & 0 & \frac{1}{7} & 0 & 0 \\ 0 & \frac{1}{7} & 0 & 0 & \frac{3}{7} & 0 & 0 & \frac{1}{7} & 0 \\ 0 & 0 & \frac{1}{7} & 0 & 0 & \frac{3}{7} & 0 & 0 & \frac{1}{7} \\ \frac{1}{21} & 0 & 0 & \frac{1}{7} & 0 & 0 & \frac{8}{21} & 0 & 0 \\ 0 & \frac{1}{21} & 0 & 0 & \frac{1}{7} & 0 & 0 & \frac{8}{21} & 0 \\ 0 & 0 & \frac{1}{21} & 0 & 0 & \frac{1}{7} & 0 & 0 & \frac{8}{21} \end{pmatrix}. \quad (89)$$